

# A statistical noise model for a class of Physically Unclonable Functions

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## Abstract

The interest in “Physically Unclonable Function”-devices has increased rapidly over the last few years, as they have several interesting properties for system security related applications like, for example, the management of cryptographic keys. Unfortunately, the output provided by these devices is noisy and needs to be corrected for these applications.

Related error correcting mechanisms are typically constructed on the basis of an equal error probability for each output bit. This assumption does not hold for Physically Unclonable Functions, where varying error probabilities can be observed. This results in a *generalized binomial distribution* for the number of errors in the output.

The intention of this paper is to discuss a novel Bayesian statistical model for the noise of an especially wide-spread class of Physically Unclonable Functions, which properly handles the varying output stability and also reflects the different noise behaviors observed in a collection of such devices. Furthermore, we compare several different methods for estimating the model parameters and apply the proposed model to concrete measurements obtained within the CODES research project in order to evaluate typical correction and stabilization approaches.

## 1 Introduction and Preliminaries

A simple but widely-used transmission model in coding theory is the Binary Symmetric Channel (BSC). The general assumptions for this transmission channel (cf. [5]) are as follows:

- bitstrings (i.e. words consisting of the symbols 0 and 1) are transmitted,
- for each symbol the “flip probability” (i.e. the probability that 0 and 1 is sent, but 1 and 0 is received, respectively) is constant over all transmissions,

- transmission errors occur independently.

Under these assumptions, the number of transmission errors in an  $n$ -bit word can be modeled as a binomially distributed random variable.

However, when investigating Physically Unclonable Functions (PUFs), these assumptions do not apply. PUFs—and in particular SRAM-PUFs—generally produce a noisy bitstring (response) for a given input (challenge) where the flipping probability varies from bit to bit. Thus, a BSC is no longer an exact model for the noise of SRAM-PUFs (cf. [2]). Another approach to model PUF error behavior (concentrating on considerations regarding entropy) is pursued in [14].

From a system security theoretic point of view, PUFs are very interesting because they can be used to construct a challenge-response mechanism without the need of having a “master key”. Usually, the master key is used to derive and verify responses to some given input challenges. However, as PUFs basically are hardware challenge-response mechanisms, a master key is not necessary. A general introduction to the topic of PUFs can be found in [12].

In our setting, an SRAM-PUF consists of  $n \in \mathbb{N}$  uninitialized SRAM cells (cf. [8]). When powering on the device, these cells either assume state 0 or 1—and most of them do so in a very stable way. Based on a series of measurements, we are able to identify a “stable state” for each cell, which, in turn, is then used to compute the *error probability* (or *flipping probability*) for each cell. For example, if a cell takes the state 1 more often than the state 0 in our measurements, we assess 1 as the stable state of this cell and 0 is its error state. A query of all  $n$  PUF bits is called a *PUF evaluation*. Otherwise, when investigating only a subset of all PUF bits, then we speak of *PUF responses*.

From a statistical point of view, the SRAM cells can be modeled by means of independent Bernoulli distributed random variables  $X_j \sim B(p_j)$  for  $j = 1, 2, \dots, n$ . That is, we have  $\mathbb{P}(X_j = 1) = p_j$  and  $\mathbb{P}(X_j = 0) = 1 - p_j$ . The stability of an SRAM-cell is then related to its error probability, where  $p_j = 0.5$  means complete instability and  $p_j = 0$  or  $p_j = 1$  means total stability. As we are only interested in the stability of the cells, and not in the concrete values they assume, our error probabilities can be bounded from above by 0.5—this can be enforced by choosing the stable state of the cell accordingly.

If these random variables are Bernoulli distributed with a common error probability  $p$ , it is well known that the random number of errors  $E$  in a response of length  $\ell$  follows a binomial distribution with parameters  $\ell$  and  $p$ , short  $E \sim \text{Bi}(\ell, p)$ . However, when investigating PUFs, the situation is not quite as simple. Not all of the SRAM cells are equally stable, which is indicated in Figure 1, showing histograms for the error rate and bit weight<sup>1</sup> for a given set of SRAM-PUF measurements. One can see that the majority of bits (about 80%) have an error rate of less than 2%, while about 2.3% of all SRAM cells show an error probability greater than 40%, indicating strong instability of these cells. Moreover, from the symmetry in the bit-weight histogram one can see that the PUF actually is very balanced: there are approximately the same number of bits assuming stable state 0 as there are assuming stable state 1.

In the case of varying error probabilities, the random number of errors no longer follows a classical binomial distribution. Instead, we apply a *generalized binomial distribution*, which yields a much better fit than the binomial distribution for the number of flipped

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<sup>1</sup>The bit weight  $w_j$  is used to determine the stable states. It is computed separately for each bit as the proportion of ones among all measurements.

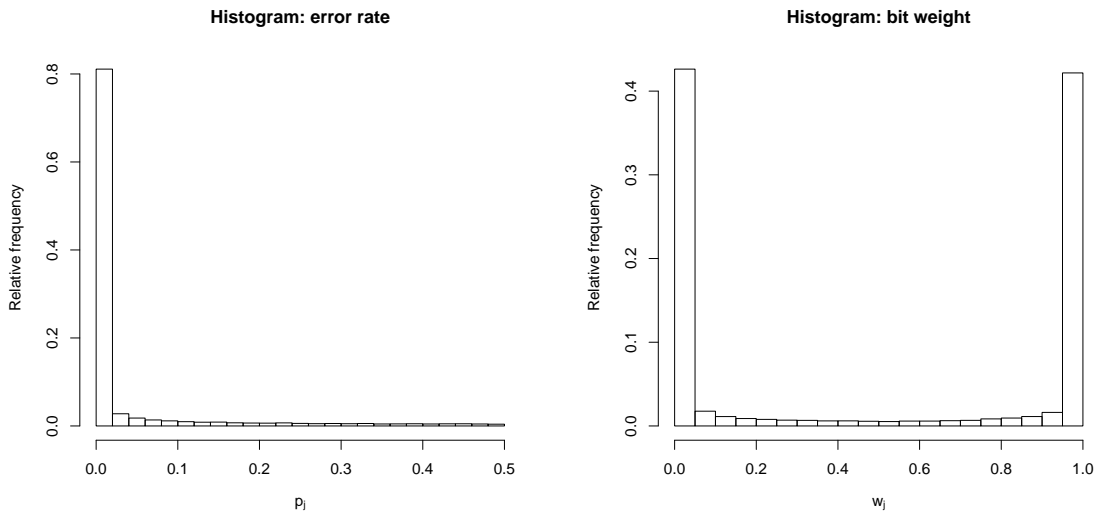


Figure 1: SRAM cell stability: bit weights and error rates.

cells per PUF evaluation. This is illustrated in Figure 2, showing a fitted binomial and generalized binomial distribution for the histogram of the number of flipped cells in a SRAM-PUF response of size  $n = 2^{16}$ .

In this paper, we investigate various properties of the generalized binomial distribution as the proper model for the noise of SRAM-PUFs (cf. Section 2). Furthermore, we develop a suitable statistical model for evaluating the overall noise behavior of SRAM-PUFs on the basis of measurements from several SRAM-devices (cf. Section 3). An empirical Bayesian approach is employed to assess the parameters of the underlying generalized binomial distribution. Several estimation techniques for determining the hyperparameters in the empirical Bayesian model are compared within a simulation study in Section 4.

Finally, we extend the proposed model to concrete measurements obtained within the CODES<sup>2</sup> research project and use these concrete measurements to discuss stabilization methods for SRAM-PUF responses on the basis of error correction schemes and order statistics.

## 2 Some properties of the generalized binomial distribution

In this section, we will discuss some properties of the generalized binomial distribution which arises naturally when investigating the number of flipped SRAM cells per SRAM-PUF response.

In general, the binomial distribution originates in the context of Bernoulli trials. In [3], the author calls the trials under which the “generalized binomial distribution” originates, *Poisson trials*<sup>3</sup>. These are  $n$  independent trials where the probability that some event occurs in the  $j$ -th trial is  $p_j$ . Then, the number of occurrences of the event within those  $n$  trials follows a generalized binomial distribution with parameter vector  $\mathbf{p} = (p_1, \dots, p_n)^\top$ . Thus, we may define this distribution as follows:

<sup>2</sup><https://www.technikon.com/projects/codes>

<sup>3</sup>Therefore, the generalized binomial distribution is also often called *Poisson binomial distribution*.

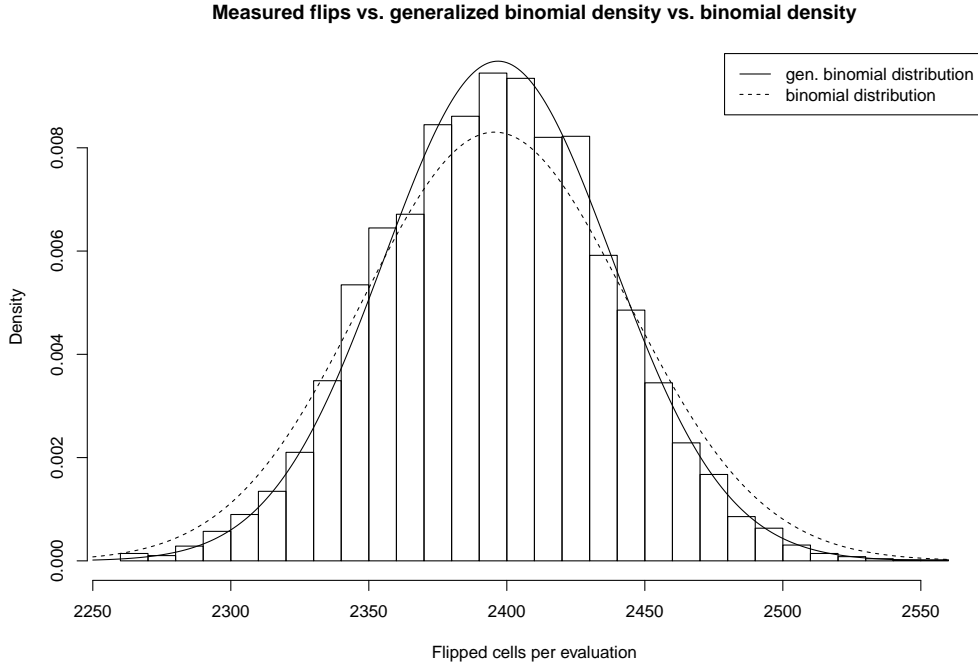


Figure 2: Goodness of fit: binomial vs. generalized binomial distribution.

**Definition 2.1 (Generalized binomial distribution).**

Let  $p_1, p_2, \dots, p_n \in [0, 1]$ . Assume that the random variables  $X_j \sim B(p_j)$  are independent for  $j \in \{1, 2, \dots, n\}$ . Then, the random variable  $X := \sum_{j=1}^n X_j$  follows a *generalized binomial distribution* with parameter vector  $\mathbf{p} = (p_1, p_2, \dots, p_n)^\top$ . For short, we write  $X \sim \text{GBi}(p_1, p_2, \dots, p_n)$ .

There is a number of elementary properties for the generalized binomial distribution which follow immediately from the definition.

**Proposition 2.1 (Elementary properties).**

Let  $X \sim \text{GBi}(p_1, p_2, \dots, p_n)$ . Then the following statements hold:

- (a) The support of the random variable  $X$  is contained in  $\{0, 1, \dots, n\}$ .
- (b) The probability mass function of  $X$  is given by

$$\mathbb{P}(X = k) = \sum_{\substack{S \subseteq \{1, 2, \dots, n\} \\ |S| = k}} \prod_{s \in S} p_s \cdot \prod_{s \in \{1, 2, \dots, n\} \setminus S} (1 - p_s).$$

- (c) Expectation and variance of  $X$  are given by

$$\mathbb{E}X = \sum_{j=1}^n p_j \quad \text{and} \quad \text{Var}(X) = \sum_{j=1}^n p_j \cdot (1 - p_j),$$

respectively.

- (d) The characteristic function  $\varphi_X(t)$  of  $X$  is given by

$$\varphi_X(t) = \prod_{j=1}^n (1 - p_j + p_j \cdot e^{it}).$$

- (e) The generalized binomial distribution is a generalization of the binomial distribution: for  $p_1 = p_2 = \dots = p_n =: p$ , the random variable  $X$  follows a binomial distribution with parameters  $n$  and  $p$ .

*Proof.* With the notation of Definition 2.1, (a) follows directly as we know  $\text{supp}(X_j) \subseteq \{0, 1\}$  for all  $j \in \{1, 2, \dots, n\}$ . Therefore, the support of the sum is contained in  $\{0, 1, \dots, n\}$ . Based on the independence of  $X_j$ , the described probability mass function as stated in (b) follows immediately from

$$\begin{aligned} \mathbb{P}(X = k) &= \sum_{\substack{S \subseteq \{1, 2, \dots, n\} \\ |S| = k}} \mathbb{P}(X_s = 1 \text{ for } s \in S \text{ and } X_s = 0 \text{ for } s \in \{1, 2, \dots, n\} \setminus S) \\ &= \sum_{\substack{S \subseteq \{1, 2, \dots, n\} \\ |S| = k}} \prod_{s \in S} p_s \cdot \prod_{s \in \{1, 2, \dots, n\} \setminus S} (1 - p_s). \end{aligned}$$

The remaining statements follow from the definition of  $X$  as the sum of independently distributed random variables. We find

$$\mathbb{E}X = \mathbb{E}\left(\sum_{j=1}^n X_j\right) = \sum_{j=1}^n \mathbb{E}X_j = \sum_{j=1}^n p_j,$$

by using the linearity of  $\mathbb{E}$ . We obtain  $\text{Var}(X_j) = p_j(1 - p_j)$  by using the linearity of the variance for independent random variables. Statement (d) is proved as the characteristic function of  $X_j$  is  $\varphi_{X_j}(t) = 1 - p_j + p_j e^{it}$  and because

$$\varphi_{X_1 + X_2 + \dots + X_n}(t) = \prod_{j=1}^n \varphi_{X_j}(t)$$

holds for independent random variables  $X_1, \dots, X_n$ . Finally, as the probabilities in (e) are all equal to  $p$ ,  $X$  is the sum of  $n$  independent and identically distributed Bernoulli random variables—which is an alternative definition for the binomial distribution with parameters  $n$  and  $p$ . This completes the proof.  $\square$

The R package “GenBinomApps” (cf. [11]) offers an efficient implementation to compute the probability mass function recursively. The theoretic background for this recursive computation approach is covered in [10].

However, for very large dimensions of the probability vector, the computation of the distribution function is quite expensive. In such cases, approximation with a binomial distribution would be desirable. In the following proposition we give some useful properties of such an approximation.

**Proposition 2.2 (Binomial approximation).**

Let  $X \sim \text{GBi}(p_1, p_2, \dots, p_n)$  and  $Y \sim \text{Bi}(n, p^*)$ . Then the following properties hold:

- (a) The expectation of  $X$  is equal to the expectation of  $Y$  if and only if  $p^* = \bar{p} := \frac{1}{n} \sum_j p_j$ .
- (b) If we have  $p^* = \bar{p}$ , then the inequality  $\text{Var}(X) \leq \text{Var}(Y)$  holds for arbitrary parameters  $p_1, p_2, \dots, p_n \in [0, 1]$  of  $X$  and equality holds if and only if  $p_1 = p_2 = \dots = p_n$ .

*Proof.* The first statement follows immediately from Proposition 2.1 and because of simple properties of the binomial distribution. We have  $\mathbb{E}X = \sum_{j=1}^n p_j$  and  $\mathbb{E}Y = n \cdot p^*$ . Therefore, the relation

$$\mathbb{E}X = \mathbb{E}Y \iff p^* = \frac{1}{n} \sum_{j=1}^n p_j$$

follows immediately. The inequality from (b) can be proven by showing its equivalency to the Cauchy-Schwarz inequality. Let  $\mathbf{1}$  denote the  $n$ -dimensional vector of ones and  $\mathbf{p} = (p_1, p_2, \dots, p_n)^\top$ . Then we obtain

$$\begin{aligned} \text{Var}(X) \leq \text{Var}(Y) &\iff \sum_{j=1}^n p_j \cdot (1 - p_j) \leq n \cdot \bar{p} \cdot (1 - \bar{p}) \\ &\iff \sum_j p_j \cdot (1 - p_j) \leq \left( \sum_j p_j \right) \cdot \left( 1 - \frac{1}{n} \sum_j p_j \right) \\ &\iff \sum_j p_j - \sum_j p_j^2 \leq \sum_j p_j - \frac{1}{n} \left( \sum_j p_j \right)^2 \\ &\iff \left( \sum_j p_j \right)^2 \leq n \cdot \sum_j p_j^2 \\ &\iff |\langle \mathbf{1}, \mathbf{p} \rangle|^2 \leq \langle \mathbf{1}, \mathbf{1} \rangle \cdot \langle \mathbf{p}, \mathbf{p} \rangle, \end{aligned}$$

which is the Cauchy-Schwarz inequality for the Euclidean scalar product. Finally, equality holds in the Cauchy-Schwarz inequality if and only if the vectors  $\mathbf{1}$  and  $\mathbf{p}$  are linearly dependent, that is  $\mathbf{p} = p \cdot \mathbf{1}$ . In this case,  $p_1 = p_2 = \dots = p_n = p$  follows, which is equivalent to the fact that  $X$  is binomially distributed with parameters  $n$  and  $p$ .  $\square$

By property (b) of Proposition 2.2, the variance of the generalized binomial distribution equals the variance of the approximating binomial distribution if and only if the distributions coincide. In the following example, we investigate whether the variance may be used to assess the goodness of fit for such an approximation.

#### **Numerical example.**

After performing some simulations (in which the input parameters of the generalized binomial distribution were generated from various beta distributions), we plotted the difference between the variances against the maximum error between the corresponding distribution functions. The result of one of these simulation batches ( $N = 1000$  generalized binomial distributions with  $n = 100$  Be(1.5, 1.8)-distributed parameters each in the left plot, and following a Be(0.3, 0.1)-distribution in the right plot) is illustrated in Figure 3. The plots depict the relation between the difference of the variances and the maximum approximation error of the respective distribution functions mentioned above.

It is interesting to see that there is a very strong correlation (with correlation coefficient greater than 0.95) between the difference of the variances and the maximal approximation error in our simulations when the  $p_j$  concentrate around a single value. In this case, the variance difference also is significantly lower (as can also be seen in Figure 3).

The following considerations require the notion of *compound distributions*.

#### **Definition 2.2 (Compound distribution).**

Let  $\mathbf{X}$  be a random vector with probability density function  $p_{\mathbf{X}}(\mathbf{x} | \boldsymbol{\theta})$  depending on some

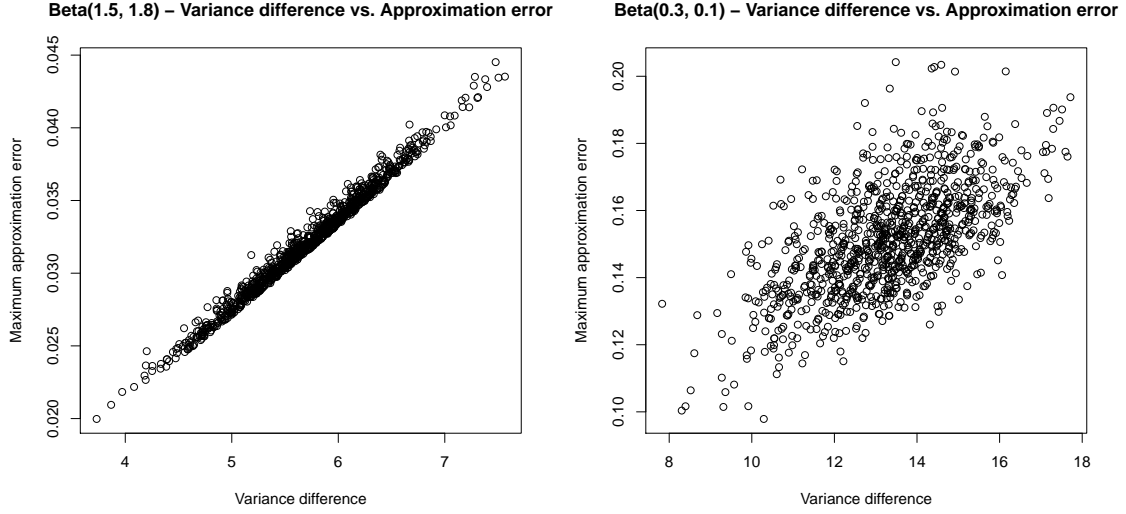


Figure 3: Binomial approximation of the generalized binomial distribution: variance vs. maximum approximation error.

random vector  $\boldsymbol{\theta}$ . Furthermore, let  $G_{\boldsymbol{\theta}}(\boldsymbol{\vartheta})$  be the distribution function of  $\boldsymbol{\theta}$ . Then the probability density of the compound distribution of  $\mathbf{X}$  with respect to  $G$  is given by

$$p_{\mathbf{X}}(\mathbf{x}) = \int_{\boldsymbol{\vartheta}} p_{\mathbf{X}}(\mathbf{x} \mid \boldsymbol{\vartheta}) dG_{\boldsymbol{\theta}}(\boldsymbol{\vartheta}).$$

Now, let us assume that the error probabilities  $p_j$  are realizations of some random variable  $\pi$  with distribution function  $F_{\pi}$  and  $\text{supp}(\pi) \subseteq [0, 1]$ . A quite interesting question is, how the compound distribution of the generalized binomial distribution with respect to the parameters of the distribution of  $\pi$  looks like. The following theorem characterizes these compound distributions with respect to the generalized binomial distribution.

**Theorem 2.3 (Compound distribution for the generalized binomial distribution).**

Let  $\pi_1, \pi_2, \dots, \pi_n$  be independently and identically distributed random variables on  $[0, 1]$  with distribution function  $F_{\pi|\boldsymbol{\vartheta}}$ . Then the compound distribution  $\text{GBi}(\pi_1, \pi_2, \dots, \pi_n)$  under the parameter vector  $\boldsymbol{\vartheta}$  is the binomial distribution  $\text{Bi}(n, \mathbb{E}(\pi \mid \boldsymbol{\vartheta}))$ .

*Proof.* It is easy to see that the expected value  $\mathbb{E}(\pi \mid \boldsymbol{\vartheta})$  always exists and is contained in  $[0, 1]$ , as the support of  $\pi$  itself is contained in  $[0, 1]$ . The expected value therefore is a valid second parameter for the binomial distribution.

Now, let  $X \mid \pi_1, \dots, \pi_n \sim \text{GBi}(\pi_1, \dots, \pi_n)$  and  $\pi_i \stackrel{\text{iid}}{\sim} F_{\pi|\boldsymbol{\vartheta}}$ . According to the definition of compound distributions, the probability mass function  $p_X(k \mid \boldsymbol{\vartheta})$  is then determined as follows:

$$\begin{aligned} p_X(k \mid \boldsymbol{\vartheta}) &= \sum_{\substack{S \subseteq \{1, \dots, n\} \\ |S|=k}} \prod_{s \in S} \left( \int_{p_s=0}^1 p_s dF_{\pi|\boldsymbol{\vartheta}} \right) \cdot \prod_{s \notin S} \left( 1 - \int_{p_s=0}^1 p_s dF_{\pi|\boldsymbol{\vartheta}} \right) \\ &= \sum_{\substack{S \subseteq \{1, \dots, n\} \\ |S|=k}} \prod_{s \in S} \mathbb{E}(\pi \mid \boldsymbol{\vartheta}) \cdot \prod_{s \notin S} (1 - \mathbb{E}(\pi \mid \boldsymbol{\vartheta})) = \binom{n}{k} \mathbb{E}(\pi \mid \boldsymbol{\vartheta})^k \cdot (1 - \mathbb{E}(\pi \mid \boldsymbol{\vartheta}))^{n-k}. \end{aligned}$$

This is exactly the probability mass function of the  $\text{Bi}(n, \mathbb{E}(\pi \mid \boldsymbol{\vartheta}))$ -distribution and therefore, proves the theorem.  $\square$

**Remark.** (i) This result is in accordance with the result stated in property (b) of Proposition 2.2: by forming the compound distribution, the variance also increases in general.

(ii) In a Bayesian context, the integrand in Definition 2.2 can be identified with the likelihood (parameterized by  $\boldsymbol{\vartheta}$ ), the integrating function  $G_{\boldsymbol{\theta}}$  with the prior distribution of  $\boldsymbol{\theta}$  and the left-hand side  $p_{\mathbf{X}}(\mathbf{x})$  portrays the marginal distribution of  $\mathbf{X}$ , also known as prior predictive distribution. The increase in the variance of the compound distribution then accounts for the parameter uncertainty in a natural way.

**Corollary 2.4 (Properties of the compound distribution).**

Let  $X \sim \text{GBi}(\pi_1, \pi_2, \dots, \pi_n)$  with the identically and independently distributed random variables  $\pi_1, \pi_2, \dots, \pi_n$  on  $[0, 1]$  with distribution function  $F_{\pi|\boldsymbol{\vartheta}}$ . Then the expected value and variance of  $X$  are given by

$$\mathbb{E}X = n \cdot \mathbb{E}(\pi \mid \boldsymbol{\vartheta}) \quad \text{and} \quad \text{Var}(X) = n \cdot \mathbb{E}(\pi \mid \boldsymbol{\vartheta}) \cdot (1 - \mathbb{E}(\pi \mid \boldsymbol{\vartheta})).$$

Furthermore, the characteristic function of  $X$  has the form

$$\varphi_X(t) = \left[ 1 - \mathbb{E}(\pi \mid \boldsymbol{\vartheta}) + \mathbb{E}(\pi \mid \boldsymbol{\vartheta}) \cdot e^{it} \right]^n.$$

*Proof.* These statements follow immediately from Theorem 2.3 and some elementary properties of the binomial distribution.  $\square$

We will use these results in Section 4 in order to estimate the probability that the SRAM-PUF noise cannot be “corrected” properly in a response of given length.

### 3 Statistical Model for overall SRAM-PUF noise behavior

In this section, we discuss a suitable statistical model for the noise behavior of a set of different SRAM-PUF devices, where the model parameters can be estimated from a series of simple PUF evaluations. The model is based on an exploratory statistical analysis carried out within the CODES research project. Furthermore, we propose several methods for assessing the model parameters.

#### 3.1 Model development

We propose a Bayesian model for the noise behavior of SRAM-PUFs: assume that we have  $m_{\text{dev}}$  SRAM-PUF devices, where device  $i$  has  $c_i$  SRAM-cells for  $i = 1, \dots, m_{\text{dev}}$ . Then, we model the noise of the  $i$ -th device as a vector of  $c_i$  independently distributed random variables  $X_{ij} \sim \text{B}(p_{ij})$  such that

$$\mathbb{P}(\text{Cell } j \text{ in device } i \text{ flips}) := \mathbb{P}(X_{ij} = 1) = p_{ij}.$$

Motivated by the results of an exploratory statistical analysis, we further model these error probabilities to follow a scaled beta distribution on the interval  $[0, 1/2]$ .



**Definition 3.1 (Scaled beta distribution).**

Let  $a$  and  $b$  be real numbers and  $a < b$ . If the random variable  $P$  follows a beta distribution with parameters  $\alpha$  and  $\beta$ ,  $P \sim \text{Be}(\alpha, \beta)$ , the random variable  $Q = a + (b - a) \cdot P$  follows a scaled beta distribution on the interval  $[a, b]$  with parameters  $\alpha$  and  $\beta$ . For short, we write  $Q \sim \text{Be}_{[a,b]}(\alpha, \beta)$ .

For the  $i$ -th device, we parametrize the beta distribution of the parameters  $(p_{ij})_{j=1}^{c_i}$  as  $\text{Be}_{[0,1/2]}(2\delta_i \cdot K_i, (1 - 2\delta_i) \cdot K_i)$ , such that  $\delta_i$  denotes the distribution's expected value and  $K_i$  is a shape parameter controlling the variance. In the next step, we assign prior distributions to  $\delta_i$  and  $K_i$ . More precisely, we model  $(\delta_i)_{i=1}^{m_{\text{dev}}}$  to follow a scaled beta distribution (again scaled to the interval  $[0, 1/2]$ ) with parameters  $\alpha$  and  $\beta$ , and the shape parameters  $(K_i)_{i=1}^{m_{\text{dev}}}$  to follow a gamma distribution with parameters  $\kappa$  and  $\lambda$ . Altogether, we have

$$\begin{aligned} X_{ij} | p_{ij} &\sim \text{B}(p_{ij}) && \text{for } i = 1, 2, \dots, m_{\text{dev}}, \quad j = 1, 2, \dots, c_i, \\ (p_{ij} | \delta_i, K_i)_{j=1}^{c_i} &\sim \text{Be}_{[0,1/2]}(2\delta_i \cdot K_i, (1 - 2\delta_i) \cdot K_i) && \text{for } i = 1, 2, \dots, m_{\text{dev}}, \\ (\delta_i | \alpha, \beta)_{i=1}^{m_{\text{dev}}} &\sim \text{Be}_{[0,1/2]}(\alpha, \beta), \\ (K_i | \kappa, \lambda)_{i=1}^{m_{\text{dev}}} &\sim \text{Ga}(\kappa, \lambda). \end{aligned}$$

**Remark.**

We choose this model over a simplified model without assumed distributions for the parameters  $\delta_i$  and  $K_i$  primarily because of the control we have over the mean error rate, as well as to reflect that different devices may have varying mean error rates. Within this model, a variety of situations related to the SRAM-PUF production can be modeled and simulated.

### 3.2 Parameter estimation

Assuming we have  $m_{\text{dev}}$  SRAM-devices with  $c_i$  SRAM-cells in the  $i$ -th device, the result of a series of  $m_i$  measurements of device  $i$  is a vector  $\mathbf{x}_i = (x_{i1}, \dots, x_{ic_i})^\top$ , where the component  $x_{ij}$  denotes the number of measured error states for the  $j$ -th cell of device  $i$  and is  $\text{Bi}(m_i, p_{ij})$ -distributed. Starting from these measurements, we wish to estimate the parameters  $\alpha$  and  $\beta$  of the scaled beta distribution  $\text{Be}_{[0,1/2]}(\alpha, \beta)$  modeling the mean error rates  $\delta_i$ , and the parameters  $\kappa$  and  $\lambda$  of the gamma distribution  $\text{Ga}(\kappa, \lambda)$  modeling the distribution of the shape parameters  $K_i$ . Note that the component  $x_{ij}$  is a realization of a  $\text{Bi}(m_i, p_{ij})$  distribution.

**Remark.**

The joint posterior density function for our model is of the form

$$\begin{aligned} f(\mathbf{p}, \boldsymbol{\delta}, \mathbf{K}, \alpha, \beta, \kappa, \lambda | \mathbf{x}) &\propto \left( \prod_{i=1}^{m_{\text{dev}}} \left( \prod_{j=1}^{c_i} f_{\text{Bi}}(x_{ij} | p_{ij}) \cdot f_{\text{Be}}(p_{ij} | \delta_i, K_i) \right) \right. \\ &\quad \left. \cdot f_{\text{Be}}(\delta_i | \alpha, \beta) \cdot f_{\text{Ga}}(K_i | \kappa, \lambda) \right) \cdot f(\alpha, \beta, \kappa, \lambda), \end{aligned}$$

where  $f_{\text{Bi}}(\cdot | p_{ij})$  denotes the density of the  $\text{Bi}(m_i, p_{ij})$  distribution, and  $f_{\text{Be}}(\cdot | \delta_i, K_i)$  and  $f_{\text{Be}}(\cdot | \alpha, \beta)$  denote the density of the  $\text{Be}_{[0,1/2]}(2\delta_i \cdot K_i, (1 - 2\delta_i) \cdot K_i)$  and  $\text{Be}_{[0,1/2]}(\alpha, \beta)$  distribution, respectively. Moreover,  $f_{\text{Ga}}(\cdot | \kappa, \lambda)$  denotes the density of the  $\text{Ga}(\kappa, \lambda)$ -distribution,  $f(\alpha, \beta, \kappa, \lambda)$  denotes some joint prior of the parameters  $\alpha$ ,  $\beta$ ,  $\kappa$  and  $\lambda$ , and

$\mathbf{x}$  denotes the vector of all measurements. Due to the (practically) very large number of parameters the simulation based on this posterior is computationally intractable.

To overcome this, we make use of an empirical Bayesian approach, meaning that we approximate the “expensive” posterior  $f(\mathbf{p}, \boldsymbol{\delta}, \mathbf{K}, \alpha, \beta, \kappa, \lambda \mid \mathbf{x})$  by the joint density  $f(\mathbf{p}, \boldsymbol{\delta}, \mathbf{K} \mid \hat{\alpha}, \hat{\beta}, \hat{\kappa}, \hat{\lambda}, \mathbf{x})$  (which is also called “pseudo posterior”) with empirically estimated parameters  $\hat{\alpha}$ ,  $\hat{\beta}$ ,  $\hat{\kappa}$  and  $\hat{\lambda}$ .

We estimate the parameters according to the model hierarchy:

- from the measurements  $x_{ij}$ , we estimate the flipping probabilities  $p_{ij}$ ,
- from the estimated flipping probabilities, we estimate the parameters  $\delta_i$  and  $K_i$  for  $i = 1, 2, \dots, m_{\text{dev}}$ ,
- and from these estimated parameters, we estimate the hyperparameters  $\alpha$ ,  $\beta$ ,  $\kappa$  and  $\lambda$ .

The estimates of the hyperparameters are then depending on the estimation techniques used for the different parameter layers. For example, possible approaches are the *method of moments*, *maximum likelihood estimation* (MLE), or the construction of a Bayes estimator.

By MLE for the  $p_{ij}$ , we obtain  $\hat{p}_{ij} = \frac{x_{ij}}{m_i}$ . In this case, this MLE-estimator coincides with the estimator obtained by the method of moments. Another approach to estimate this parameter (in the context of Bayesian statistics) is to choose the expected value of the posterior obtained with respect to a scaled Jeffreys prior,  $p_{ij} \sim \text{Be}_{[0,1/2]}(1/2, 1/2)$ . The posterior distribution is then given by

$$f(p_{ij} \mid x_{ij}) \propto \frac{p_{ij}^{x_{ij}} (1 - p_{ij})^{m_i - x_{ij}}}{\sqrt{(2p_{ij})(1 - 2p_{ij})}},$$

forcing us to compute the expectation numerically as

$$\mathbb{E}(p_{ij} \mid x_{ij}) = \hat{p}_{ij} = \int_0^{1/2} p_{ij} \cdot f(p_{ij} \mid x_{ij}) dp_{ij},$$

or to approximate it by Monte Carlo simulation. An advantage of this approach for the estimation of the flipping probabilities is that it avoids an underestimation of the flipping probabilities in the zero error case. This is because the method of moments and MLE yield a flipping probability of 0 for cells without observed errors, which is not realistic. Moreover, this approach allows us to compute sensible credible intervals for these probabilities, whereas the usual confidence intervals based on the MLE would have zero lengths and thus be meaningless.

Given the estimates of  $p_{i1}, \dots, p_{ic_i}$ , we can estimate the parameters  $\delta_i$  and  $K_i$ , either again by MLE, by the method of moments or by a Bayesian approach similar to the one above, where we use the joint noninformative prior

$$p(\delta, K) \propto \frac{1}{K \cdot \sqrt{(2\delta)(1 - 2\delta)}}.$$

After estimating the  $\delta_i$  and  $K_i$ , we may use these values to obtain estimations for the hyperparameters  $\alpha$ ,  $\beta$ ,  $\kappa$  and  $\lambda$ . As we want to avoid proposing more priors for these parameters, we will use either MLE or the method of moments.

## 4 Results and Applications

On the basis of the statistical model proposed in the previous section, we will determine a suitable method for parameter estimation of this model in Section 4.1 by comparing possible approaches in a simulation study. Afterwards, in Section 4.2, we will estimate the parameters of our model (based on the superior estimation method from the simulation study) for real measurements from the CODES project. Finally, in Section 4.3, we will use the posterior predictive distribution based on our real data to evaluate approaches to correct and stabilize the SRAM-PUF responses.

### 4.1 Simulation study

We are interested in comparing different parameter estimation methods as discussed in Section 3.2 for the proposed statistical model. In order to choose the “best” estimation method, we will estimate these hyperparameters from simulated data with known hyperparameters. The quality of these estimation methods will then be compared by the value of a quadratic loss function for the parameter vector:  $L(\theta, \hat{\theta}) = \|\theta - \hat{\theta}\|^2$ , where  $\theta \in \{(\alpha, \beta)^\top, (\kappa, \lambda)^\top\}$ . Note that we are especially interested in a good estimation of the parameters  $\alpha$  and  $\beta$  of the beta distribution modeling the mean error rates.

We will generate the simulation data ( $m_{\text{dev}} = 20$  SRAM-devices with  $c = 10000$  cells each and  $m = 500$  simulations per device) from the following parameters<sup>4</sup>:

$$\alpha = 100, \quad \beta = 900, \quad \kappa = 800, \quad \lambda = 900.$$

Concretely, there are 8 methods of parameter estimation we will compare. These methods originate from the different possibilities to estimate the various parameter hierarchies. Let  $x_{ij}$  denote the number of assumed error states of the  $j$ -th cell in device  $i$ .

- The flipping probabilities  $p_{ij}$  can be estimated by the method of moments (which, in this case, coincides with maximum likelihood estimation) by  $\hat{p}_{ij} = \frac{x_{ij}}{m}$ , or by computing the Bayes-estimator with respect to the Jeffreys  $\text{Be}_{[0,1/2]}(1/2, 1/2)$ -prior.
- The parameters  $\delta_i$  and  $K_i$  can be estimated either by the method of moments, yielding the estimators

$$\hat{\delta}_i = \frac{1}{c} \sum_{j=1}^c \hat{p}_{ij}, \quad \hat{K}_i = \frac{2\hat{\delta}_i(1 - 2\hat{\delta}_i)}{\frac{4}{c-1} \cdot \sum_{j=1}^c (\hat{p}_{ij} - \hat{\delta}_i)^2} - 1,$$

by maximum likelihood estimation with the R-package `maxLik` (cf. [6]), or by using a Bayesian approach and computing the mode of the joint (independence) posterior distribution subject to the Jeffreys  $\text{Be}_{[0,1/2]}(1/2, 1/2)$ -prior for  $\delta_i$  and the non-informative  $\frac{1}{\delta}$ -prior for  $K_i$ .

- Finally, the hyperparameters  $\alpha, \beta$  from the proposed beta distribution of the  $\delta_i$  and the parameters  $\kappa, \lambda$  from the proposed gamma distribution of the  $K_i$  can be estimated by the method of moments, which yields

$$\hat{\alpha} = 2\bar{\delta} \left( \frac{2\bar{\delta}(1 - 2\bar{\delta})}{4v_{\delta}} - 1 \right), \quad \hat{\beta} = (1 - 2\bar{\delta}) \left( \frac{2\bar{\delta}(1 - 2\bar{\delta})}{4v_{\delta}} - 1 \right), \quad \hat{\kappa} = \frac{\bar{K}^2}{v_K}, \quad \hat{\lambda} = \frac{\bar{K}}{v_K},$$

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<sup>4</sup>These parameters are roughly based on parameters we used for testing in the CODES project.

where  $\bar{\delta}$ ,  $v_{\delta}$ ,  $\bar{K}$  and  $v_K$  denote the means and sample variances of the  $\hat{\delta}_i$  and the  $\hat{K}_i$ , respectively—or by maximum likelihood estimation. As we want to refrain from proposing another set of priors for these parameters, we will not use Bayesian estimation for  $\alpha$ ,  $\beta$ ,  $\kappa$  and  $\lambda$ .

In order to compare the various possible combinations of parameter estimation methods, we used a quadratic loss function to measure the distance from the original parameters. After performing 10000 simulations, and investigating the respective mean losses (which can be found in Table 1), we find that estimating the  $p_{ij}$  and the parameters  $\delta_i$  and  $K_i$  with Bayesian methods, as well as the hyperparameters  $\alpha$ ,  $\beta$ ,  $\kappa$  and  $\lambda$  with maximum likelihood estimation yields the lowest overall loss (where the estimates of all four hyperparameters are taken into account) as well as the lowest loss for just the parameters  $\alpha$  and  $\beta$  of the beta distribution. However, the lowest loss for the parameters  $\kappa$  and  $\lambda$  of the gamma distribution originates from estimating the  $p_{ij}$  with Bayesian methods, but using the method of moments to estimate everything else.

Note that even although the approach where we estimate  $p_{ij}$  as well as  $\delta_i$  and  $K_i$  by Bayesian means, and the remaining parameters by MLE yields the lowest loss function with respect to the parameters  $\alpha$  and  $\beta$ , this estimator is rather conservative with respect to the expected mean error rate  $\mathbb{E}\delta = \frac{1}{2} \cdot \frac{\alpha}{\alpha+\beta} = 0.05$ : taking the average value over the  $\frac{1}{2} \cdot \frac{\hat{\alpha}}{\hat{\alpha}+\hat{\beta}}$  obtained in the simulation study (where  $\hat{\alpha}$  and  $\hat{\beta}$  have been constructed as mentioned above) yields a value of 0.0618, which can be contributed to the high number of observed very unstable bits. In order to cover the occurrence of such bits, the expected average error rate is increased in the MLE-estimation. Therefore, estimating the parameters with this approach yields a model, which possesses a certain “robustness” regarding a decline of the PUF’s stability. The most accurate approximation of the expected mean error rate is obtained by estimating all parameters by the method of moments.

$\hat{p}_{ij}$	$\hat{\delta}_i, \hat{K}_i$	$\hat{\alpha}, \hat{\beta}, \hat{\kappa}, \hat{\lambda}$	Mean loss ( $\alpha, \beta$ )	Mean loss ( $\kappa, \lambda$ )
Moments/MLE	Moments	Moments	10684999.5	202115.0
Moments/MLE	MLE	MLE	6573128.9	397729.0
Bayes	Moments	Moments	11077285.5	<u>200906.9</u>
Bayes	Moments	MLE	22138561.9	236938.5
Bayes	MLE	Moments	15936758.9	917774.9
Bayes	MLE	MLE	7831878.6	750779.0
Bayes	Bayes	Moments	14402261.7	968651.6
<b>Bayes</b>	<b>Bayes</b>	<b>MLE</b>	<u>3042068.4</u>	1895868.2

Table 1: Average loss function values from the simulation study (10000 simulations).

## 4.2 Parameter estimation for real data

We are investigating measurements originating from  $m_{\text{dev}} = 15$  different SRAM-PUF devices, each of them with  $c = 2^{16}$  SRAM cells. Note that our given measurements were carried out on ASICs that have been manufactured in TSMC 65nm CMOS technology within a European multi-project wafer run. The ASIC has been designed within the UNIQUE<sup>5</sup> research project.

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<sup>5</sup><http://www.unique-project.eu>

For each device, we have 340 evaluations—however, the first 50 measurements are discarded because they were conducted during an aging process, which slightly changed the behavior of the SRAM-PUFs. Afterwards, during the remaining  $m = 290$  measurements, the devices are stable again, meaning that we will focus our analysis on these measurements.

For the parameter estimation, we will follow the results of the simulation study, meaning that we will estimate the flipping probabilities  $p_{ij}$  and the parameters  $\delta_i$  and  $K_i$  by the Bayesian approaches described above, and the four parameters  $\alpha$ ,  $\beta$ ,  $\kappa$  and  $\lambda$  by maximum likelihood estimation.

This results in the following estimates:

$$\hat{\alpha} = 9378.324 \quad \hat{\beta} = 81409.79, \quad \hat{\kappa} = 7166.669, \quad \hat{\lambda} = 3965.296.$$

In Figure 4, we plotted histograms for the (respectively) estimated  $\delta_i$  and  $K_i$ , as well as the densities of the proposed probability distributions.

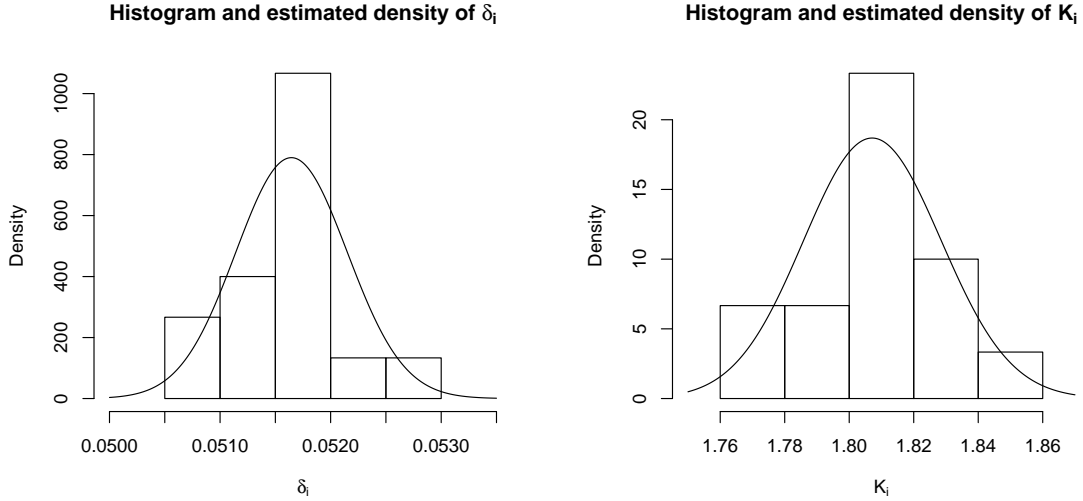


Figure 4: Histograms and estimated densities for  $\delta_i$  and  $K_i$ .

Furthermore, we may compute the expected values for the parameters  $\delta$  and  $K$ , and construct credible intervals. From the estimated parameters we obtain

$$\mathbb{E}\delta = \frac{1}{2} \frac{\alpha}{\alpha + \beta} \approx 0.05165, \quad \mathbb{E}K = \frac{\kappa}{\lambda} \approx 1.8073,$$

and empirical 95% credible intervals

$$\delta \in [0.05066, 0.05264], \quad K \in [1.76574, 1.84943].$$

### 4.3 Error correction and reduction

In practice, we are interested in stabilizing the responses of a PUF such that it can be used for system security related aspects like constructing a challenge-response system without the need of storing a master key. We want to present two general approaches

(concentrating on error correction and error reduction) for stabilizing these responses and use our proposed model in order to evaluate their effectiveness in specific examples.

Based on the parameters  $\alpha$ ,  $\beta$ ,  $\kappa$  and  $\lambda$  estimated in the previous section, we may investigate the posterior-predictive distribution for the flipping probabilities. Its density function is given by

$$f(p \mid \hat{\alpha}, \hat{\beta}, \hat{\kappa}, \hat{\lambda}, \mathbf{x}) = \int_{\delta} \int_K f_{\text{Be}}(p \mid \delta, K) \cdot f_{\text{Be}}(\delta \mid \hat{\alpha}, \hat{\beta}) \cdot f_{\text{Ga}}(K \mid \hat{\kappa}, \hat{\lambda}) dK d\delta$$

for  $p \in (0, 1/2)$  and 0 otherwise. By simulation of a sample of size 100000, we obtain  $\bar{p} = 0.05187$  as an approximation for the expected value of the posterior-predictive distribution under the empirically estimated parameters  $\hat{\alpha}$ ,  $\hat{\beta}$ ,  $\hat{\kappa}$  and  $\hat{\lambda}$  from above.

We are interested in the number of errors in an  $\ell$ -bit SRAM-PUF response, where the bit-wise error probabilities are distributed according to the posterior-predictive distribution from above. For fixed error probabilities, the quantity of errors follows a generalized binomial distribution. Due to Theorem 2.3, the resulting compound distribution is a  $\text{Bi}(\ell, \bar{p})$ -distribution. This distribution can now be used to compute the probability that a given error correction mechanism fails.

#### Numerical example.

Assume that an SRAM-PUF is embedded within a construction which allows the correction of up to 239 bits in responses of length  $\ell = 1953$ . Following the model above, the expected number of errors is 101.3101, and the probability that the PUF does not work properly (i.e. the probability that more than 239 errors occur) is negligibly small (less than  $10^{-20}$ ).

Instead of designing powerful mechanisms for error correction which are able to compensate for the noise an SRAM-PUF produces, another approach is to “ignore” SRAM-cells for which a high flipping probability is known or has been estimated. Assuming that we are investigating  $\ell$  bit SRAM-PUF responses, it is an interesting question how the removal of  $r \ll \ell$  unstable bits influences the noise behavior.

In general, a good measure to judge the effect of ignoring the  $r$  “worst” bits is the average flipping probability of the remaining cells, that is if  $p_1, \dots, p_{\ell}$  are the respective flipping probabilities, and  $p_{(1)} \leq \dots \leq p_{(\ell)}$  denote the related ordered probabilities, then we are interested in  $\frac{1}{\ell-r} \cdot \sum_{j=1}^{\ell-r} p_{(j)}$ . The assumption that these probabilities are realizations of identical and independently distributed random variables leads us to *order statistics*.

#### Definition 4.1 (Order statistics).

Let  $X_1, \dots, X_n$  be identical and independently distributed random variables with respect to some distribution  $X$ . Then the *ordered* random variables  $X_{(k)}$  with  $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$  are called  $k$ -th smallest *order statistic* of size  $n$  with respect to  $X$ .

The following theorem states a central result from the theory of order statistics, a proof can be found in [1].

#### Theorem 4.1 (Order statistics and the beta distribution).

The density function of the  $k$ -th smallest order statistic  $U_{(k)}$  of size  $n$  with respect to the uniform distribution on the interval  $[0, 1]$  is

$$f_{U_{(k)}}(u) = \begin{cases} \frac{n!}{(k-1)!(n-k)!} \cdot u^{k-1} \cdot (1-u)^{n-k} & \text{for } u \in (0, 1), \\ 0 & \text{else,} \end{cases}$$

which is the density of a beta distribution with parameters  $k$  and  $n - k + 1$ . Therefore, we have  $U_{(k)} \sim \text{Be}(k, n - k + 1)$ .

By the technique of *Probability Integral Transform*, this result may be used to express the density function of an arbitrary continuous random variable  $X$  with distribution function  $F_X$  and density function  $f_X$ : in this case, we obtain

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} \cdot [F_X(x)]^{k-1} \cdot [1 - F_X(x)]^{n-k} \cdot f_X(x)$$

for the density function of the  $k$ -th smallest order statistic of size  $n$  with respect to the distribution of  $X$ .

**Remark.**

Note that as for  $1 \leq j < k \leq n$  the relation  $X_{(j)} \leq X_{(k)}$  holds, the order statistics are *not* independently distributed any more. This means that if we would like to compute some compound distribution of, for example, the generalized binomial distribution and these order statistics with respect to a (scaled) beta distribution (which could be used to predict the probability that an  $\ell$ -bit PUF response with the  $r$  most unstable bits removed is still too noisy for correction), we would have to consider the respective joint densities (which can be found in [1]) when integrating over the respective parameters. As we focus on the scaled beta distribution, the arising integrals cannot be computed analytically (mainly because of the occurring products of incomplete beta functions)—however, by simulating the procedure, i.e. generating  $N$  beta-distributed samples of size  $\ell$  and removing the  $r$  largest values, we may still give good approximations for some interesting parameters within specific examples.

In the setting above, we are interested in the expected flipping probability, averaged over all remaining SRAM-cells, that is

$$\mathbb{E} \left[ \frac{1}{\ell - r} \cdot \sum_{j=1}^{\ell-r} \pi_{(j)} \right] = \frac{1}{\ell - r} \cdot \sum_{j=1}^{\ell-r} \mathbb{E} \pi_{(j)}.$$

Thus, we primarily want to compute the expected value of the order statistics. For the same reasons as mentioned in the remark above, this is not possible analytically for arbitrary distributions. However, for a special case of the scaled beta distribution it is actually possible, and that is for  $\text{Be}(\alpha, \beta)$  with  $\alpha = 1$  or  $\beta = 1$ .

**Proposition 4.2.**

Let  $\pi_1, \dots, \pi_n$  be independent and identically distributed random variables following a  $\text{Be}_{[a,b]}(\alpha, 1)$ -distribution. The expected value of the  $k$ -th smallest order statistic of size  $n$  is then given by

$$\mathbb{E} \pi_{(k)} = a + (b - a) \cdot \frac{B(k + 1/\alpha, n - k + 1)}{B(k, n - k + 1)}.$$

*Proof.* As scaled beta distributed random variables are affine-linearly transformed beta distributed random variables (which also holds for the related order statistics), and as the expectation is a linear operator, we may concentrate on the case  $\pi_1, \dots, \pi_n \stackrel{\text{iid}}{\sim} \text{Be}(\alpha, 1)$ .

The density function of these random variables is given by  $f(x) = \mathbb{1}_{(0,1)}(x) \cdot \alpha \cdot x^{\alpha-1}$ , and

thus the distribution function has the shape

$$F(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ x^\alpha & \text{for } 0 < x < 1, \\ 1 & \text{for } x \geq 1. \end{cases}$$

As mentioned above, by *Probability Integral Transform*, the density of the  $k$ -th smallest order statistic of size  $n$  with respect to the  $\text{Be}(\alpha, 1)$ -distribution has the form

$$\begin{aligned} f_{\pi_{(k)}}(x) &= k \binom{n}{k} \cdot [F(x)]^{k-1} \cdot [1 - F(x)]^{n-k} \cdot f(x) \\ &= \mathbb{1}_{(0,1)}(x) \cdot \alpha k \binom{n}{k} \cdot x^{\alpha k - 1} \cdot (1 - x^\alpha)^{n-k}. \end{aligned}$$

The expected value thus reads

$$\mathbb{E}\pi_{(k)} = \alpha k \binom{n}{k} \cdot \int_0^1 x^{\alpha k} (1 - x^\alpha)^{n-k} dx,$$

which, after a change of variables  $t = x^\alpha$ , becomes

$$\mathbb{E}\pi_{(k)} = \frac{\int_0^1 t^{k+1/\alpha-1} (1-t)^{n-k} dt}{B(k, n-k+1)} = \frac{B(k+1/\alpha, n-k+1)}{B(k, n-k+1)}.$$

Finally, by the transformation  $\pi_{(k)} \mapsto a + (b-a) \cdot \pi_{(k)}$ , the statement is proven.  $\square$

**Remark.**

An analogous statement holds for the  $\text{Be}_{[a,b]}(1, \beta)$ -distribution. This follows directly from the fact that if  $X$  follows a  $\text{Be}_{[a,b]}(\alpha, \beta)$ -distribution, then the linearly transformed variable  $Y = a + b - X$  follows a  $\text{Be}_{[a,b]}(\beta, \alpha)$ -distribution.

**Numerical example.**

We want to investigate responses of length  $n = 16$  of an SRAM-PUF embedded within an error correction scheme such that up to 3 errors can be corrected. For the sake of simplicity, we will assume that the cell-wise error probabilities are distributed according to a  $\text{Be}_{[0,1/2]}(1/9, 1)$ -distribution (such that the mean error rate is  $\frac{1}{2} \cdot \frac{1/9}{1/9+1} = 0.05$ ). Note that by “ignoring” bits of the PUF responses, also the error correction scheme gets weakened: for every 2 ignored bits, the correction capacity reduces by 1. Table 2 contains the expected values of the respective order statistics (computed along the lines of Proposition 4.2). Furthermore, by simulation we are able to estimate the probability that a system failure

$k$	1	2	3	4	5	6	7	8
$\mathbb{E}P_{(k)}$	$2.45 \cdot 10^{-7}$	$2.45 \cdot 10^{-6}$	$1.35 \cdot 10^{-5}$	$5.38 \cdot 10^{-5}$	0.00017	0.00048	0.00122	0.00279
$k$	9	10	11	12	13	14	15	16
$\mathbb{E}P_{(k)}$	0.00594	0.01189	0.02260	0.04110	0.07193	0.12173	0.2	0.32

Table 2: Order statistics – expected values (simplified model).

(i.e. more errors than the correction scheme can handle) occurs when ignoring the  $r$  most unstable cells. The results of this simulation (with 100000 simulated PUFs) can be found in Table 3.



Ignored cells	0	1	2	3	4	5	6
Correction capacity	3	3	2	2	1	1	0
avg. sys. failure prob.	0.00704	0.00183	0.00462	0.00134	0.00579	0.00201	0.02203
max. sys. failure prob.	0.49622	0.35504	0.49410	0.34375	0.52352	0.38755	0.67622

Table 3: System failure probabilities (simplified model).

However, practically, the simplified model is not as precise as the statistical model developed in the previous section. Therefore, we investigate a similar example based on this more sophisticated model from the previous section. In this case, all parameters will have to be estimated by simulation.

### Numerical example.

We use the posterior-predictive distribution discussed at the beginning of this section to obtain flipping probabilities for the simulated SRAM-PUFs with 512 cells each. We have plotted the results of this simulation (with 100000 simulated SRAM devices) in Figure 5. Remarkably, the mean error rate can be reduced very quickly from slightly above 0.05 to about 0.026 by ignoring the 50 most unstable bits per device (about 10% information loss). This demonstrates that the exclusion of unstable bits is a viable and practically relevant approach to increase the stability of an SRAM-PUF.

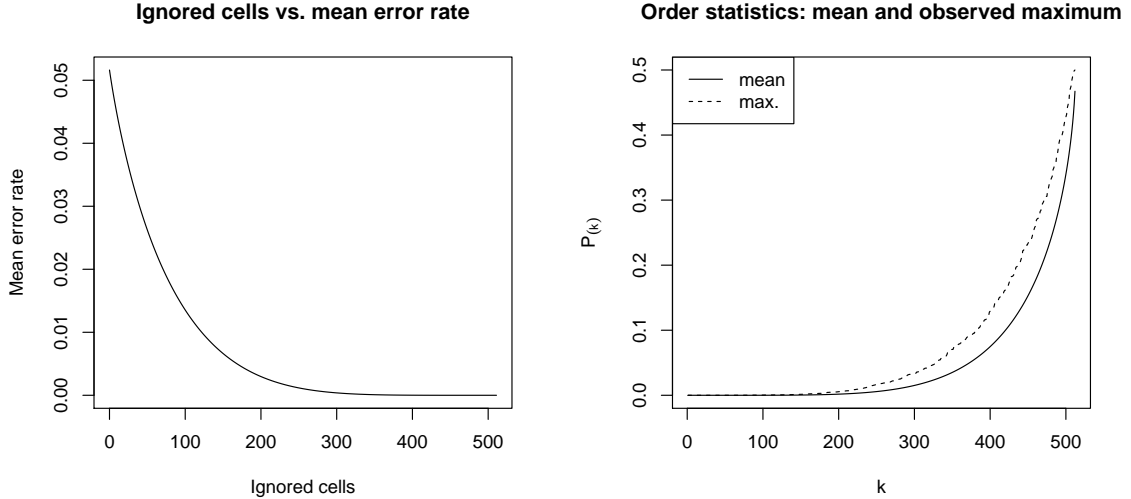


Figure 5: Order statistics and mean error rate.

## 5 Conclusion

The design of error correcting mechanisms capable of correcting the noise emitted by SRAM-PUFs requires a precise statistical analysis. In this paper, we presented the framework for such an analysis by proposing a statistical model which captures the noise behavior of a collection of SRAM-PUFs (cf. Section 3 and Section 4.2). In practice, such a model allows for a certain flexibility when designing and conducting statistical tests in the context of quality assurance—which is very important, as these tests are very expensive in general.

The second tool developed in this paper in order to ascertain precise predictions for the

number of errors in SRAM-PUF responses is the generalized binomial distribution (cf. Section 2). This distribution, in combination with the posterior-predictive distribution for the error probabilities obtained from our given measurements, permits an evaluation and the design of sufficiently strong error correction mechanisms (cf. Section 4.3). Finally, we showed that by ignoring the most unstable parts of SRAM-PUF responses, the mean error rate could be reduced significantly. Thus, weaker and simpler error correction mechanisms could be used.

## References

- [1] Mohammad Ahsanullah, Valery B. Nevzorov, and Mohammad Shakil. *An Introduction to Order Statistics*, volume 3 of *Atlantis Studies in Probability and Statistics*. Atlantis Press, 2013.
- [2] Christoph Bösch, Jorge Guajardo, Ahmad-Reza Sadeghi, Jamshid Shokrollahi, and Pim Tuyls. Efficient Helper Data Key Extractor on FPGAs. In Elisabeth Oswald and Pankaj Rohatgi, editors, *Cryptographic Hardware and Embedded Systems – CHES 2008*, volume 5154 of *Lecture Notes in Computer Science*, pages 181–197. Springer Berlin Heidelberg, 2008.
- [3] Marek Fisz. *Probability theory and mathematical statistics*. Wiley Publications in Statistics. John Wiley & Sons, Inc., 1965.
- [4] Michael S. Hamada, Alyson G. Wilson, C. Shane Reese, and Harry F. Martz. *Bayesian Reliability*. Springer Series in Statistics. Springer, 2008.
- [5] Darrel R. Hankerson, Dean G. Hoffman, Douglas A. Leonard, Charles C. Lindner, Kevin T. Phelps, Chris A. Rodger, and James R. Wall. *Coding Theory And Cryptography—The Essentials*. Chapman & Hall/CRC Pure and Applied Mathematics. CRC Press, 2000.
- [6] Arne Henningsen and Ott Toomet. maxLik: A package for maximum likelihood estimation in R. *Computational Statistics*, 26(3):443–458, 2011.
- [7] Peter D. Hoff. *A First Course in Bayesian Statistical Methods*. Springer Texts in Statistics. Springer, 2009.
- [8] Daniel E. Holcomb, Wayne P. Burleson, and Kevin Fu. Initial SRAM state as a fingerprint and source of true random numbers for RFID tags. In *Proceedings of the Conference on RFID Security*, 2007.
- [9] Karl J. Klauer. *Kriteriumsorientierte Tests: Lehrbuch der Theorie und Praxis lehrzielorientierten Messens*. Hogrefe, 1987.
- [10] Daniel Kurz, Horst Lewitschnig, and Jürgen Pilz. Decision-theoretical model for failures which are tackled by countermeasures. *IEEE Transactions on Reliability*, 63(2):583–592, 2014.
- [11] Horst Lewitschnig and David Lenzi. *GenBinomApps: Clopper-Pearson Confidence Interval and Generalized Binomial Distribution*, 2014. R package version 1.0-2.
- [12] Roel Maes and Ingrid Verbauwhede. Physically Unclonable Functions: A Study on the State of the Art and Future Research Directions. In Ahmad-Reza Sadeghi

and David Naccache, editors, *Towards Hardware-Intrinsic Security: Foundations and Practice*, Information Security and Cryptography, pages 3–37. Springer, 2010.

- [13] R Core Team. *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria, 2014.
- [14] Robbert van den Berg, Boris Skoric, and Vincent van der Leest. Bias-based Modeling and Entropy Analysis of PUFs. In *Proceedings of the 3rd International Workshop on Trustworthy Embedded Devices*, TrustED '13, pages 13–20, New York, NY, USA, 2013. ACM.